Formulation of an equation of diffusion for heterogeneous rods

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Abstract

A new formulation for predicting the energy flow in heterogeneous rods is developed on the basis of an equation of diffusion. It succeeds in surpassing traditional studies for homogeneous ones, for which vibratory responses are often easily known by simple displacement formulation. However, one stays difficult to describe the mechanical behavior for heterogeneous structures: in this case, the energy flow is governed by structural damping, as for homogeneous medium, but by discontinuities too. The study presented here proposes to identify for heterogeneous structures a factor of diffusion involving proportionality between transmitted power and energy gradient. An equation of diffusion is then formulated considering an energetic balance. The formulation is semi-local, that is spatial average is locally made to take into account several heterogeneities in studied element. An analytical formulation is first developed for rod with area discontinuities. The proposed theoretical factor of diffusion is in good agreements with the numerical predictions.

1 Introduction

Vibratory behavior of structures is often difficult to predict. In the middle and high frequency range, standard finite element method is quickly limited because element dimensions depend on wavelengths. Concerning homogeneous structures, energy formulations succeed in identifying mean and local vibratory response even at high frequency. Statistical energy Analysis (SEA) [1] is successfully used to characterize mean energies of lightly coupled subsystems; difficulties however occur to identify boundary conditions for coupled elements. The energy flow method is based on the analogy between heat conduction given by Fourier's law [2] and energy propagation, assuming that the active transmitted power is related to the energy gradient by a factor of diffusion. According to this hypothesis, this method establishes a local equation of diffusion characterizing the energy flow in the structure. Rigorously, the analogy with a Fourier's law cannot be obtained locally for a structure. An approximate form can be derived when quantities are spatially averaged over half a structural wavelength and when evanescent field is neglected [3-6]. An approximate equation of diffusion is then obtained, difficulties however occur when spatially averaged energetic boundary conditions have to be specified, moreover the assumption that the near field is neglected becomes erroneous. Therefore when heterogeneities occur, the energy flow method is unsuitable to predict vibratory behavior. It stays only applicable for homogeneous simple structures, for which classical displacement formulations are often sufficient to identify the vibratory responses.

The proposed study is concerned by energetic diffusion in heterogeneous structures. The aim is to show how diffusion phenomena occur between elements containing a random spatial distribution of heterogeneities. As for homogeneous structures, diffusion is governed by structural damping, it stays however strongly dependent on discontinuities. Formulation is semi-local as energy flow is studied between heterogeneous elements. It is approximately local when element dimensions stay negligible compared to the structural wavelength.

A rod with cross-section discontinuities is studied in this paper, diffusion is analyzed between element containing random spatial distribution of heterogeneities. A factor of diffusion is identified when element dimensions stay negligible compared to wavelength. The theoretical result are validated by a numerical simulation for a clamped-free heterogeneous rod excited at the free end by a power source.

2 Theoretical development

Let us consider the heterogeneous rod illustrated on Figure 1, composed by N elements, each one containing n_p homogeneous rods of random lengths $l_{p,i}$ and areas $S_{p,i}$, the index p,i meaning homogeneous rod i from element p. The mean potential energy density for element p is $<\overline{U}>_p$.

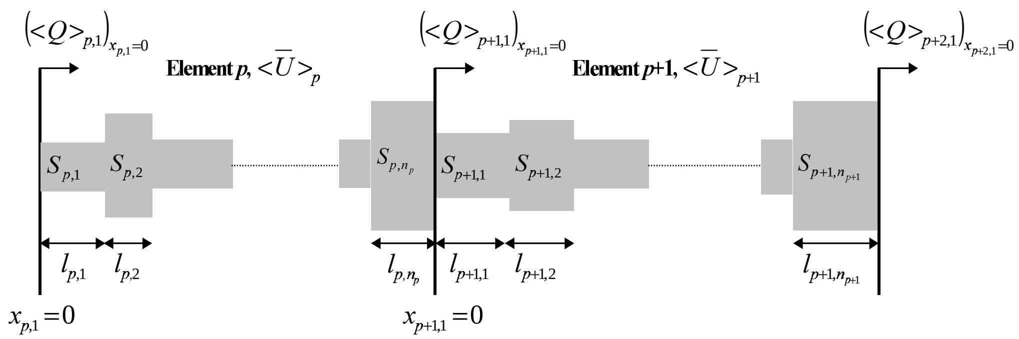

Figure 1: Description of two consecutive elements for random scheme.

The active transmitted power in homogeneous rod i from element p is expressed

$$< Q >_{p,i} = \frac{1}{2}\mathrm{Re}\left\{ \mathrm{j}\omega E S_{p,i} \frac{\partial u_{p,i}}{\partial x} u^{*}_{p,i} \right\}, \tag{1}$$

where $u_{p,i}$ is the displacement related to equation of motion for homogeneous rod outside of the external force points

$$\frac{\partial^2 u_{p,i}}{\partial x^2} + k^2 u_{p,i} = 0\,, \tag{2}$$

involving the complex wavenumber [4]

$$k = \sqrt{\omega^2 \frac{\rho}{E_R (1+\mathrm{j}\eta)}} = \frac{\omega}{c_l \sqrt{1+\mathrm{j}\eta}}\,, \tag{3}$$

when hysteretic damping is considered. c_l is the wave velocity, E_R is the real part of the Young’s modulus and η is the loss factor. Assuming η<<1, the complex wavenumber is simply expressed by

$$k \cong k_R + \mathrm{j}k_I = k_R (1 - \mathrm{j}\frac{\eta}{2})\,, \tag{4}$$

where $k_R = \omega / c_l$ and k_I are respectively the real part and the imaginary part of the complex wavenumber.
Between homogeneous rods i and i-1 from element p, continuity conditions in displacement

$$\left(u_{p,i}\right)_{x_{p,i}=0} = \left(u_{p,i-1}\right)_{x_{p,i-1}=l_{p,i-1}} \tag{5}$$

and axial force

$$S_{p,i}\left(\frac{\partial u_{p,i}}{\partial x}\right)_{x_{p,i}=0} = S_{p,i-1}\left(\frac{\partial u_{p,i-1}}{\partial x}\right)_{x_{p,i-1}=l_{p,i-1}} \tag{6}$$

allow us to precise the continuity condition for power

$$\left(< Q >_{p,i}\right)_{x_{p,i}=0} = \left(< Q >_{p,i-1}\right)_{x_{p,i-1}=l_{p,i-1}}\,, \tag{7}$$

and second derivative of the power

$$\left(\frac{\partial^2}{\partial x^2}<Q>_{p,i}\right)_{x_{p,i}=0}=\left(\frac{\partial^2}{\partial x^2}<Q>_{p,i-1}\right)_{x_{p,i-1}=l_{p,i-1}}, \tag{8}$$

if one derives equation (1) twice. Discontinuity condition for potential energy density is too precised

$$\left(\frac{\partial}{\partial x}<Q>_{p,i}\right)_{x_{p,i}=0}=\frac{S_{p,i-1}}{S_{p,i}}\left(\frac{\partial}{\partial x}<Q>_{p,i-1}\right)_{x_{p,i-1}=l_{p,i-1}}, \tag{9}$$

if one considers the energy balance for steady state and outside of the input power points [4]

$$\frac{\partial}{\partial x}<Q>_{p,i}=-2\omega\eta<U>_{p,i} \tag{10}$$

establishing that spatial derivative of the power $<Q>_{p,i}$ is related to the local potential energy density $<U>_{p,i}$.

According to equation (7) and assuming $k_R L_p << 1$ where L_p is the length of element *p*, i.e. element dimension stays negligible compared to wavelength, the variation of active power between elements *p* and *p*+1 is expressed by a first order Taylor's series,

$$\left(<Q>_{p+1,1}\right)_{x_{p+1,1}=0}=\left(<Q>_{p,n_p}\right)_{x_{p,n_p}=0}+l_{p,n_p}\left(\frac{\partial}{\partial x}<Q>_{p,n_p}\right)_{x_{p,n_p}=0}.$$

from power transmitted between rods n_p-1 and n_p from element *p*. According to equation (9),

$$\left(\frac{\partial<Q>_{p,n_p}}{\partial x}\right)_{x_{p,n_p}=0}=\frac{S_{p,1}}{S_{p,n_p}}\left(\frac{\partial}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0},$$

and the power is finally expressed

$$\left(<Q>_{p+1,1}\right)_{x_{p+1,1}=0}=\left(<Q>_{p,1}\right)_{x_{p,1}=0}+\left(\sum_{r=1}^{n_p}l_{p,r}\frac{S_{p,1}}{S_{p,r}}\right)\left(\frac{\partial}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0}, \tag{11}$$

from power $\left(<Q>_{p,1}\right)_{x_{p,1}=0}$ exchanged between elements *p*-1 and *p*. In a similar manner, the spatial derivative of the power is expressed

$$\left(\frac{\partial}{\partial x}<Q>_{p+1,1}\right)_{x_{p+1,1}=0}=\frac{S_{p,n_p}}{S_{p+1,1}}\left(\frac{\partial}{\partial x}<Q>_{p,n_p}\right)_{x_{p,n_p}=0}+\frac{S_{p,n_p}}{S_{p+1,1}}l_{p,n_p}\left(\frac{\partial^2}{\partial x^2}<Q>_{p,n_p}\right)_{x_{p,n_p}=0}.$$

Equation (8) allows us to write

$$\left(\frac{\partial^2}{\partial x^2}<Q>_{p,n_p}\right)_{x_{p,n_p}=0}=\left(\frac{\partial^2}{\partial x^2}<Q>_{p,1}\right)_{x_{p,1}=0},$$

i.e. the second derivative of the power is constant on element p, hence

$$\begin{aligned}\left(\frac{\partial}{\partial x}<Q>_{p+1,1}\right)_{x_{p+1,1}=0}&=\frac{S_{p,1}}{S_{p+1,1}}\left(\frac{\partial}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0}\\&+\left(\sum_{r=1}^{n_p}l_{p,r}\frac{S_{p,r}}{S_{p+1,1}}\right)\left(\frac{\partial^2}{\partial x^2}<Q>_{p,1}\right)_{x_{p,1}=0}\end{aligned}. \qquad (12)$$

Let us define the semi-local derivative of the power evaluated at junction between element p-1 and p

$$\left(\frac{\overline{\partial}}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0}=\frac{1}{L_p}\left[\left(<Q>_{p+1,1}\right)_{x_{p+1,1}=0}-\left(<Q>_{p,1}\right)_{x_{p,1}=0}\right], \qquad (13)$$

expressing the ratio of the difference between transmitted powers at element p boundaries to the length. According to equation (11), the semi-local derivative is

$$\left(\frac{\overline{\partial}}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0}=\frac{1}{L_p}\left(\sum_{r=1}^{n_p}l_{p,r}\frac{S_{p,1}}{S_{p,r}}\right)\left(\frac{\partial}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0}. \qquad (14)$$

from derivative of the power evaluated in element p at the coupling between element p-1 and p.

The second semi-local derivative is also defined at junction between element p-1 and p from semi-local derivatives,

$$\left(\frac{\overline{\partial^2}}{\partial x^2}<Q>_{p,1}\right)_{x_{p,1}=0}=\frac{1}{L_p}\left[\left(\frac{\overline{\partial}}{\partial x}<Q>_{p+1,1}\right)_{x_{p+1,1}=0}-\left(\frac{\overline{\partial}}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0}\right] \qquad (15)$$

that is

$$\left(\overline{\frac{\partial^2}{\partial x^2}} < Q >_{p,1}\right)_{x_{p,1}=0} = \frac{1}{L_p}\left[\frac{1}{L_{p+1}}\left(\sum_{r=1}^{n_{p+1}} l_{p+1,r}\frac{S_{p+1,1}}{S_{p+1,r}}\right)\left(\frac{\partial}{\partial x} < Q >_{p+1,1}\right)_{x_{p+1,1}=0} - \frac{1}{L_p}\left(\sum_{r=1}^{n_p} l_{p,r}\frac{S_{p,1}}{S_{p,r}}\right)\left(\frac{\partial}{\partial x} < Q >_{p,1}\right)_{x_{p,1}=0}\right]$$

and

$$\left(\overline{\frac{\partial^2}{\partial x^2}} < Q >_{p,1}\right)_{x_{p,1}=0} = \frac{1}{L_p}\left[\frac{1}{L_{p+1}}\left(\sum_{r=1}^{n_{p+1}}\frac{l_{p+1,r}}{S_{p+1,r}}\right)\left(\sum_{r=1}^{n_p} l_{p,r}S_{p,r}\right)\left(\frac{\partial^2}{\partial x^2} < Q >_{p,1}\right)_{x_{p,1}=0} + S_{p,1}\left\{\frac{1}{L_{p+1}}\left(\sum_{r=1}^{n_{p+1}}\frac{l_{p+1,r}}{S_{p+1,r}}\right) - \frac{1}{L_p}\left(\sum_{r=1}^{n_p}\frac{l_{p,r}}{S_{p,r}}\right)\right\}\left(\frac{\partial}{\partial x} < Q >_{p,1}\right)_{x_{p,1}=0}\right]. \quad (16)$$

Let us neglect the second term of the right member of equation (16), involving

$$\frac{1}{L_{p+1}}\left(\sum_{r=1}^{n_{p+1}}\frac{l_{p+1,r}}{S_{p+1,r}}\right) \cong \frac{1}{L_p}\left(\sum_{r=1}^{n_p}\frac{l_{p,r}}{S_{p,r}}\right),$$

that is the heterogeneity field slightly fluctuates between two consecutive elements (strictly verified hypothesis for similar elements), the equation for second semi-local derivative of the power is simply rewritten

$$\left(\overline{\frac{\partial^2}{\partial x^2}} < Q >_{p,1}\right)_{x_{p,1}=0} = \frac{1}{L_p^2}\left(\sum_{r=1}^{n_p}\frac{l_{p,r}}{S_{p,r}}\right)\left(\sum_{r=1}^{n_p} l_{p,r}S_{p,r}\right)\left(\frac{\partial^2}{\partial x^2} < Q >_{p,1}\right)_{x_{p,1}=0}, \quad (17)$$

from second derivative of the power evaluated at the coupling between element p-1 and p.

The mean potential energy density for element p is defined

$$< \overline{U} >_p = \frac{1}{L_p}\sum_{i=1}^{n_p}\int_{\text{rod}\, i} < U >_{p,i}\,\mathrm{d}x\,.$$

According to equation (10), it is expressed

$$< \overline{U} >_p = -\frac{1}{2\omega\eta}\left(\overline{\frac{\partial}{\partial x}} < Q >_{p,1}\right)_{x_{p,1}=0} \quad (18)$$

from the semi-local derivative of the power. The semi-local energy gradient is defined in the direction x

$$\left(\overline{\nabla}_x < \overline{U} >_p\right)_{x_{p,1}=0} = \frac{1}{L_p}[< \overline{U} >_{p+1} - < \overline{U} >_p], \tag{19}$$

from mean potential energy densities, hence

$$\left(\overline{\nabla}_x < \overline{U} >_p\right)_{x_{p,1}=0} = -\frac{1}{2\omega\eta}\left(\overline{\frac{\partial^2}{\partial x^2}} < Q >_{p,1}\right)_{x_{p,1}=0},$$

that is

$$\left(\overline{\nabla}_x < \overline{U} >_p\right)_{x_{p,1}=0} = -\frac{1}{2\omega\eta L_p^2}\left(\sum_{r=1}^{n_p}\frac{l_{p,r}}{S_{p,r}}\right)\left(\sum_{r=1}^{n_p} l_{p,r} S_{p,r}\right)\left(\frac{\partial^2}{\partial x^2} < Q >_{p,1}\right)_{x_{p,1}=0}. \tag{20}$$

The factor of diffusion for heterogeneous rod is defined at the coupling point between elements p-1 and p as follows

$$\left(D_r^{het}\right)_{x_{p,1}=0} = -\frac{\left(< Q >_{p,1}\right)_{x_{p,1}=0}}{(\overline{\nabla}_x < \overline{U} >_p)_{x_{p,1}=0}}. \tag{21}$$

It characterizes for a fixed energy gradient the power that allows to be transmitted between two consecutive elements. According to equation (20), the factor of diffusion for heterogeneous rod is expressed

$$\left(D_r^{het}\right)_{x_{p,1}=0} = \frac{2\omega\eta L_p^2}{\left(\sum_{r=1}^{n_p}\frac{l_{p,r}}{S_{p,r}}\right)\left(\sum_{r=1}^{n_p} l_{p,r} S_{p,r}\right)}\frac{\left(< Q >_{p,1}\right)_{x_{p,1}=0}}{\left(\frac{\partial^2}{\partial x^2} < Q >_{p,1}\right)_{x_{p,1}=0}} \tag{22}$$

from the factor of diffusion

$$\left(D_r^{hom}\right)_{x_{p,1}=0} = 2\omega\eta\frac{\left(< Q >_{p,1}\right)_{x_{p,1}=0}}{\left(\frac{\partial^2}{\partial x^2} < Q >_{p,1}\right)_{x_{p,1}=0}} \tag{23}$$

evaluated for homogeneous rod. Finally an equation of diffusion for heterogeneous rod is expressed outside of the input power points

$$\left(\overline{\nabla^2}_x < \overline{U} >_p\right)_{x_{p,1}=0} - \frac{2\omega\eta}{(D_r^{het})_{x_{p,1}=0}} < \overline{U} >_p = 0 \tag{24}$$

where $\overline{\nabla_x^2} = \overline{\frac{\partial}{\partial x}} \nabla_x$.
Simply if one supposes

$$\frac{\left(< Q >_{p,1}\right)_{x_{p,1}=0}}{\left(\frac{\partial^2}{\partial x^2} < Q >_{p,1}\right)_{x_{p,1}=0}} = \frac{\left(< Q >_{1,1}\right)_{x_{1,1}=0}}{\left(\frac{\partial^2}{\partial x^2} < Q >_{1,1}\right)_{x_{1,1}=0}}$$

that is the ratio of the power to its second derivative stays constant and expressed from boundary known quantities, an approximate factor of diffusion is

$$\left(D_r^{het}\right)_{x_{p,1}=0}^{ap} = \frac{2\omega\eta L_p^2}{\left(\sum_{r=1}^{n_p} \frac{l_{p,r}}{S_{p,r}}\right)\left(\sum_{r=1}^{n_p} l_{p,r} S_{p,r}\right)} \frac{\left(< Q >_{1,1}\right)_{x_{1,1}=0}}{\left(\frac{\partial^2}{\partial x^2} < Q >_{1,1}\right)_{x_{1,1}=0}}. \tag{25}$$

3 Numerical results

For example, let us consider the heterogeneous rod illustrated on Figure 2, it is clamped at one of its end and excited on the other by a known input power $\left(<Q>_{1,1}\right)_{x_{1,1}=0} = <Q>_{in}$.

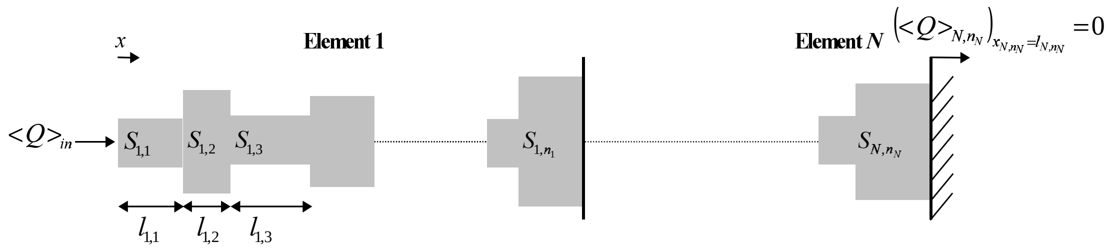

Figure 2: Clamped heterogeneous rod excited by input power $<Q>_{in}$.

It is composed by N=400 elements, each of them containing n=4 homogeneous rods. Material properties are the Young's modulus $E_R = 2.1e11Pa$, the loss factor $\eta = 0.01$ and the density $\rho = 7800kg/m^3$, areas and lengths of homogeneous rods randomly fluctuate. The structure is excited at frequency $f = 500Hz$. The exact numerical value of the factor of diffusion is compared with the exact theoretical prediction given by equation (22) and the approximate one given by equation (25). Similar elements are first considered, the results are plotted on Figure 3.

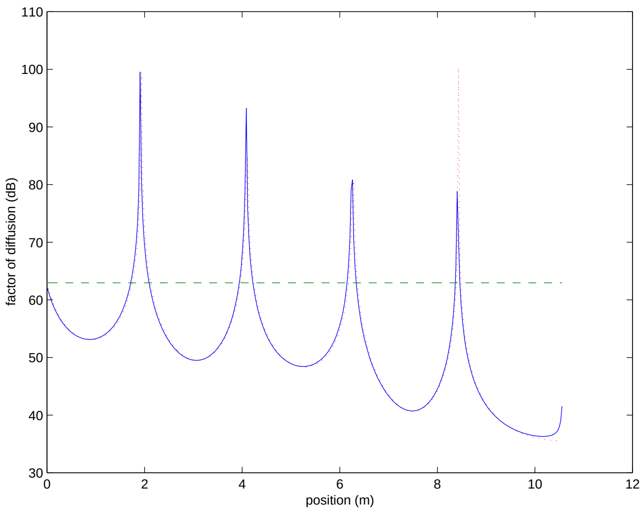

Figure 3: Factor of diffusion for similar elements, numerical value (—), exact theoretical value (···), approximate theoretical value (---).

The exact numerical value of the factor of diffusion is well described by the exact theoretical value given by equation (22). One observes located peaks introducing high power flow phenomena, periodically repeated when similar element scheme is considered. The numerical result is globally averaged by the approximate theoretical value given by equation (25) when boundary quantities are considered; it is 10 dB overestimated around the clamped end.

Random element scheme is also studied, the value of the factor of diffusion is plotted on Figure 4.

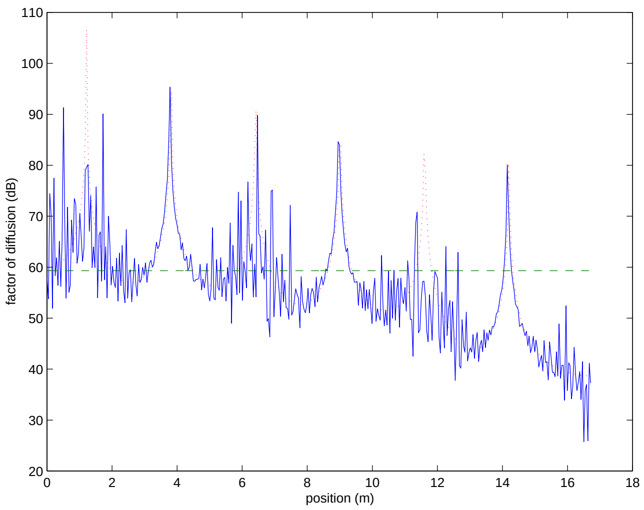

Figure 4: Factor of diffusion for random elements, numerical value (—), exact theoretical value (···), approximate theoretical value (---).

As previous case for similar elements, the theoretical value is in good agreement with the numerical one. The hypothesis supposing low variation of the heterogeneity field between two consecutive elements is validated when the exact second semi-local derivative provided by equation (16) is compared with the approximate one given by equation (17) on Figure 5.

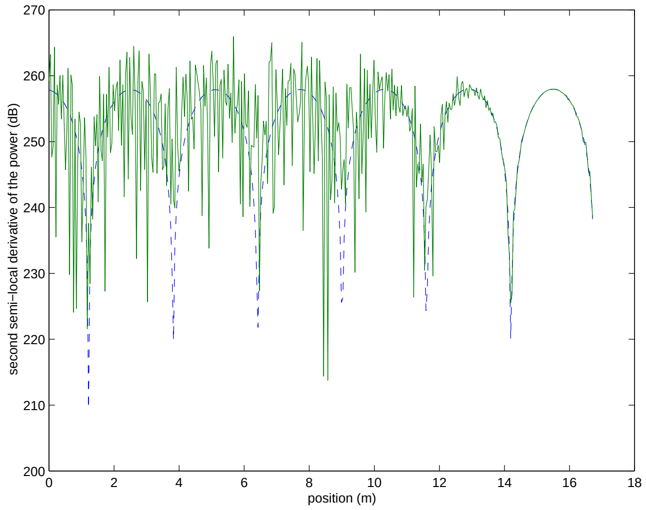

Figure 5: Second semi-local derivative of the power, exact theoretical value (—), approximate theoretical value (---).

As supposed, the considered hypothesis is validated as the exact value of the second semi-local derivative well fits with the approximate one.

4 Conclusion

In this work, a new formulation characterizing the energy flow in heterogeneous rod was derived when heterogeneities are modeled by area discontinuities. This method is semi-local when energetic diffusion is identified between elements containing a random scheme of heterogeneities. The calculated factor of diffusion depends on structural damping as for homogeneous structure, it is also governed by area discontinuities. The theoretical results succeed in estimating the exact value provided by the numerical simulation when the factor of diffusion is spatially plotted along the length of the rod.

A next important development in analyzing the energetic diffusion in heterogeneous structure is to study more complicated structures, such as beams with area discontinuities and plates with located masses. Final aim is to predict vibratory behavior for industrial case like car frame. It involves another important study when identifying energetic boundary conditions for heterogeneous coupled subsystems, that is local transmitted powers and energies.

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